ON FLAT BUNDLES IN CHARACTERISTIC 0 AND $p > 0$

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Abstract. We discuss analogies between the fundamental groups of flat bundles in characteristic 0 and $p > 0$.

1. Introduction

In this short note, we discuss some analogies between $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules over X quasi-projective over $k = \mathbb{C}$, the field of complex numbers, and over an algebraically closed field of characteristic $p > 0$. For the finiteness problems we singled out, they are striking. Yet we have no way to go from characteristic 0 to characteristic $p > 0$ and vice-versa.

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2. Characteristic 0

Let X be a smooth connected quasi-projective variety defined over the field $\mathbb{C}$ of complex numbers, together with a complex point $a \in X(\mathbb{C})$. Then one has the topological fundamental group $\pi_1^{\rm top}(X, a)$ of loops centered at a of the underlying topological spaces, modulo homotopy, as defined by Poincaré ([28]).

Grothendieck developed a different viewpoint on it. He considered the category of topological covers of X. The objects are topological covers $\pi : Y \to X$ while the maps $\theta : \pi_1 \to \pi_2$ are continuous maps $Y_1 \to Y_2$ over X. The point a defines a fiber functor $\omega_a^{\rm top}$ to the category of sets by sending π to $\pi^{-1}(a)$ and θ to $\theta|_a : \pi_1^{-1}(a) \to \pi_2^{-1}(a)$. Then the fundamental group becomes identified with the automorphism group of $\omega_a^{\rm top}$: $\pi_1^{\rm top}(X, a) \xrightarrow{\cong} \mathrm{Aut}(\omega_a^{\rm top})$, by sending a loop centered at a to the collection of induced automorphisms of $\pi^{-1}(a)$ for all topological covers π ([6, 10.11]).

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He restricted $\omega_a^{\rm top}$ to the full subcategory of finite covers of X. By the Riemann existence theorem, the natural functor from the category of finite étale covers of X to the category of finite topological covers is an equivalence of categories. Restricting $\omega_a^{\rm top}$ to it defines the fiber functor ω_a from the category of finite étale covers of X to finite sets. This defines the étale fundamental group $\pi_1^{\rm \acute{e}t}(X,a)$ centered at a as the automorphism group of ω_a. This is thus the profinite completion of $\pi_1^{\rm top}(X,a)$. Grothendieck's definition has the advantage to be general. He defines the étale fundamental group $\pi_1^{\rm \acute{e}t}(X,a)$ in [30, V] for any connected normal scheme X exactly in the same way, in particular for $X = \operatorname{Spec} k$, where k is a field, in which case $\pi_1^{\rm \acute{e}t}(X,a)$ becomes identified with $\mathrm{Gal}(\bar{k}/k)$ where $\bar{k} \subset \kappa(a)$ is the separable closure of k lying in the residue field of the chosen geometric point. The fundamental theorem in those theories says that the category of topological (resp. finite étale) covers of X is equivalent, via the fiber functor $\omega_a^{\rm top}$ (resp. ω_a), to the representation category of $\pi_1^{\rm top}(X,a)$ (resp. $\pi_1^{\rm \acute{e}t}(X,a)$) in the category of sets (resp. finite sets). ([6], *loc. cit.* [30, V, Proposition 5.8]).

Coming back to X a smooth connected quasi-projective variety over $\mathbb{C}$, the underlying topological space is a finite CW complex ([2, Remark p.40]), and thus $\pi_1^{\rm top}(X,a)$ is finitely presented, in particular finitely generated. This implies that its pro-algebraic completion

$$\pi_1^{\rm top}(X,a)^{\rm alg} := \varprojlim_H H$$

where $H \subset GL(r,\mathbb{C})$ is the Zariski closure of a complex linear representation of $\pi_1^{\rm top}(X,a)$, is controlled by the pro-finite completion $\pi_1^{\rm \acute{e}t}(X,a)$. For example, if $f: X \to Y,\ a \mapsto b$ is a morphism of smooth complex connected quasi-projective varieties, then if $f_*: \pi_1^{\rm \acute{e}t}(X,a) \to \pi_1^{\rm \acute{e}t}(Y,b)$ is an isomorphism, so is $f_*: \pi_1^{\rm top}(X,a)^{\rm alg} \to \pi_1^{\rm top}(Y,b)^{\rm alg}$ (Malcev [24], Grothendieck [14, Théorème 1.2]). In particular, applied to $Y = \operatorname{Spec}\mathbb{C}$, it says that if the étale fundamental group is trivial, there are no non-trivial complex linear representations of $\pi_1^{\rm top}(X,a)$, that is no non-trivial complex linear systems. The proof of this last point is easy. Since $\pi_1^{\rm top}(X,a)$ is finitely generated, a complex linear representation $\rho: \pi_1^{\rm top}(X,a) \to GL(r,\mathbb{C})$ has values in $GL(r,A)$ where A is a ring of finite type over $\mathbb{Z}[\frac{1}{N}]$, for some natural number $N \neq 0$. If ρ is not trivial, then there is a closed point $s \in \operatorname{Spec} A$, with finite residue field $\kappa(s)$, such that the specialization $\rho \otimes \kappa(s): \pi_1^{\rm top}(X,a) \to GL(r,\kappa(s))$ is not trivial as well.

Let us write the three groups in a diagram. One has

$$\begin{tikzcd} \pi_1(X,a)^{\rm top} \arrow[d] \arrow[dr] & \\ \pi_1^{\rm top}(X,a)^{\rm alg} \arrow[r] & \pi_1^{\rm \acute{e}t}(X,a) \end{tikzcd} \tag{2.1}$$

where the top group is a finitely presented abstract group, the left bottom group is a pro-algebraic group over $\mathbb{C}$, the right bottom group is a pro-finite group, which is the pro-finite completion of the top one, and the horizontal map is a morphism of pro-algebraic groups over $\mathbb{C}$ when one thinks of the bottom right group as a constant pro-algebraic group over $\mathbb{C}$. The horizontal map is surjective as any finite étale cover $\pi : Y \to X$ defines the local system $\pi_*\mathbb{C}$ with finite monodromy. In fact, the horizontal map is the pro-finite completion map. We saw that in the sense discussed above, $\pi_1^{\text{ét}}(X, a)$ controls $\pi_1^{\text{top}}(X, a)^{\text{alg}}$. So $\pi_1^{\text{top}}(X, a)^{\text{alg}}$ is controlled by its profinite completion.

On the other hand, by the Riemann-Hilbert correspondence [5], the representation category of $\pi_1^{\text{top}}(X, a)^{\text{alg}}$ in finite dimensional complex vector spaces is tensor equivalent to the category of vector bundles with a regular singular flat connection. The latter is equivalent to the category of regular singular $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules. This notion is purely algebraic in the following sense: if k is a field of characteristic 0, X is a smooth geometrically connected quasi-projective variety defined over k, then the category of vector bundles (E, ∇) with a flat connection, or equivalently the category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules, is an abelian, k-linear, rigid tensor category, neutralized by the choice of a k-point $a \in X(k)$. The fiber functor ω_a^{alg} sends (E, ∇) to $E|_a$. The Tannaka group $\pi_1^{\text{alg}}(X, a) := \text{Aut}^{\otimes}(\omega_a^{\text{alg}})$ is then a pro-algebraic group over k, its representation category in finite dimensional k vector spaces is via ω_a^{alg} equivalent to the category of vector bundles with a flat connection. The category of flat connections contains the full subcategory of flat connections with regular singularities at infinity [5]. Denoting by $\omega_a^{\text{alg,rs}}$ the restriction of ω_a^{alg} to this subcategory defines the Tannaka group $\pi_1^{\text{alg,rs}}(X, a) := \text{Aut}^{\otimes}(\omega_a^{\text{alg,rs}})$ as a quotient of $\pi_1^{\text{alg}}(X, a)$. If $k = \mathbb{C}$, then $\pi_1^{\text{top}}(X, a)^{\text{alg}} = \pi_1^{\text{alg,rs}}(X, a)$.

Again considering $\pi_1^{\text{ét}}(X, a)$ as a constant pro-algebraic group over k, the homomorphism

$$\pi_1^{\text{alg}}(X, a) \to \pi_1^{\text{ét}}(X, a) \tag{2.2}$$

factors through

$$\pi_1^{\text{alg}}(X, a) \to \pi_1^{\text{alg,rs}}(X, a) \to \pi_1^{\text{ét}}(X, a) \tag{2.3}$$

and $\pi_1^{\text{ét}}(X, a)$ is the pro-finite completion both of $\pi_1^{\text{alg}}(X, a)$ and $\pi_1^{\text{alg,rs}}(X, a)$.

However, if $K \supset k$ is a field extension, the natural base change morphisms

$$\pi_1^{\text{alg}}(X \otimes_k K, a \otimes_k K) \to \pi_1^{\text{alg}}(X, a) \otimes_k K$$

and

$$\pi_1^{\text{alg,rs}}(X \otimes_k K, a \otimes_k K) \to \pi_1^{\text{alg,rs}}(X, a) \otimes_k K$$

of pro-algebraic groups over K are not isomorphisms ([6, 10.35]). It is discussed in *loc. cit.* over $X = \mathbb{G}_m$, but this is still not an isomorphism even if X is

projective, in which case the surjective homomorphism $\pi_1^{\rm alg}(X,a) \to \pi_1^{\rm alg,rs}(X,a)$ is an isomorphism. For example, assume $K = \overline{k(t)}$, $H^0(X, \Omega^1_X) \neq 0$. Then the flat connection $(\mathcal{O}_X, d + t\alpha)$, for $0 \neq \alpha \in H^0(X, \Omega^1_X)$ can not be a subquotient of a connection defined over k. Still the base change morphisms are faithfully flat, as any subconnection $(E', \nabla') \subset (E, \nabla) \otimes_k K$ is defined over a finite extension of $L \supset k$ in K ([7, Proposition 2.21]).

Proposition 2.1 ([14], Théorème 1.2 over $\mathbb{C}$)**.** *Let k be an algebraically closed field of characteristic 0.*

i) *If $f: X \to Y$ is a morphism of smooth connected quasi-projective varieties mapping $a \in X(k)$ to $b \in Y(k)$. If $f_*: \pi_1^{\rm \acute{e}t}(X,a) \to \pi_1^{\rm \acute{e}t}(Y,b)$ is an isomorphism, then $f_*: \pi_1^{\rm alg,rs}(X,a) \to \pi_1^{\rm alg,rs}(Y,b)$ is an isomorphism as well.*
ii) *If X is a smooth connected quasi-projective variety with $a \in X(k)$, such that $\pi_1^{\rm \acute{e}t}(X,a)$ is trivial. Then $\pi_1^{\rm alg,rs}(X,a)$ is trivial as well.*

Proof. Then if $k \subset K$ is an extension of algebraically closed fields, the base change morphism

$$\pi_1^{\rm \acute{e}t}(X \otimes_k K, a \otimes_k K) \to \pi_1^{\rm \acute{e}t}(X,a) \tag{2.4}$$

is an isomorphism ([20, Introduction]).

We show i). Assuming k is embeddable in $\mathbb{C}$, the assumption implies that $\pi_1^{\rm alg,rs}(X \otimes_k \mathbb{C}, a \otimes_k \mathbb{C}) \to \pi_1^{\rm alg,rs}(Y \otimes_k \mathbb{C}, b \otimes_k \mathbb{C})$ is an isomorphism by Malcev-Grothendieck's theorem. So given M a flat regular singular connection on Y, and $N \subset f^*M$ a subconnection (thus regular singular) on X, there is a subconnection $N' \subset M \otimes_k \mathbb{C}$ (thus regular singular, as $\otimes \mathbb{C}$ preserves the property) with $(f \otimes_k \mathbb{C})^*(N') = N \otimes_k \mathbb{C}$. These relations are defined over an affine variety S over k with $k(S) \subset \mathbb{C}$. Thus if $p_1^Z : Z \times_k S \to Z$ denotes the first projection, one obtains $N'_S \subset (p_1^Y)^*M$ with $(f \times 1_S)^*(N'_S) = (p_1^X)^*N$ for some relative flat connection N_S. Choosing $s \in S(k)$ a rational point, the restriction N'_s of N'_S to $Y \times_k s$ fulfills $N'_s \subset M$ (thus is regular singular) and $f^*(N'_s) = N$. This shows that $\pi_1^{\rm alg,rs}(X,a) \to \pi_1^{\rm alg,rs}(Y,b)$ is faithfully flat ([7, Proposition 2.21]).

Similarly, if M is a flat regular singular connection on X, there is a flat regular singular connection N on $Y \otimes_k \mathbb{C}$ such that M is a subquotient of $(f \otimes \mathbb{C})^*(N)$. As N is regular singular, there is a smooth compactification $\hat{Y}$ of Y such that $\hat{Y}$ is projective and $D = \hat{Y} \setminus Y$ is a normal crossings divisor, and there is a locally free extension $\hat{N}$ of N such that the connection extends to $\hat{N} \to \Omega^1_{\hat{Y} \otimes_k \mathbb{C}/\mathbb{C}}(\log D) \otimes \hat{N}$. Again spreading out, $\hat{N}$ is obtained by base change from a flat connection connection $\hat{N}_S \to \Omega^1_{\hat{Y} \times_k S/S}(\log D) \otimes \hat{N}_S$ on $Y \times_k S$ relative to S. So restricting $N_S = \hat{N}_S|_{Y \times_k S}$ to a k point of S shows that M is a subquotient of a flat connection f^*N_0, with N_0 a flat regular singular connection defined over Y. This shows that

$\pi_1^{\mathrm{alg,rs}}(X,a) \to \pi_1^{\mathrm{alg,rs}}(Y,b)$ is a closed embedding ([7], *loc. cit.*). This finishes the proof if k is embeddable in $\mathbb{C}$.

In general, an object or a morphism between two objects is defined over a field K_0 of finite type over $\mathbb{Q}$ in k containing the field of definition k_0 of f, so $f = f_0 \otimes_{k_0} k$. One applies the previous isomorphism to an algebraic closure $\bar{K}_0$ of K_0 to conclude that over $\bar{K}_0$, $(f_0 \otimes_{k_0} \bar{K}_0)^*$ identifies objects and morphisms. This finishes the proof of i).

ii) is a particular case of i) for $Y = \operatorname{Spec} k$.

□

Proposition 2.1 i), while applied to f being the Albanese map, implies

Corollary 2.2 (see [13], Theorem 0.4 over $\mathbb{C}$)**.** *Let k be an algebraically closed field of characteristic* 0*, let X be a smooth projective variety over k. Then $\pi_1^{\text{ét}}(X,a)$ is abelian, if and only if every irreducible bundle with a flat connection has rank* 1*.*

Proposition 2.1 is an algebraic statement. Its proof relies on the theorem of Malcev-Grothendieck, which in turn is a consequence of $\pi_1^{\mathrm{top}}(X,a)$ being finitely generated for X defined over $\mathbb{C}$. We are not aware of the existence of an algebraic proof to it.

To close this section, we observe that the finite generation of $\pi_1^{\mathrm{top}}(X,a)$ is also reflected, for X projective smooth over $\mathbb{C}$, by the moduli theory of Simpson [29]. The Betti moduli space $M_B(X)$, which parametrizes complex local systems of rank r, is a complex affine variety. He constructed the de Rham moduli space $M_{dR}(X)$, and, fixing a polarization, the Higgs moduli space $M_{\mathrm{Higgs}}(X)$. Both are quasi-projective varieties. They map to the moduli spaces of semi-stable bundles, so his construction via geometric invariant theory generalizes the classical one for vector bundles.

Furthermore, they are all homeomorphic, in fact even real analytically isomomorphic. So for example, if one knows that $M_{\mathrm{Higgs}}(X)$ is projective, then necessarily it is 0-dimensional, that is there are finitely many isomorphism classes of irreducible rank r local systems (see [19] where this argument is used). But to conclude directly Proposition 2.1 ii), one would need for instance that points in $M_{dR}(X)$ say, which correspond to local systems with finite monodromy, are dense. It is true rank 1. In higher rank there are rigid local systems which are isolated (we thank Burt Totaro for the reference [4]) . More generally, it would be of high interest to understand the points in the three moduli spaces which, in $M_{\mathrm{Betti}}(X)$, correspond to local systems with finite monodromy. Those flat connections (E,∇) are uniquely determined by the underlying vector bundle E. By Nori [25], such vector bundles (which he called "finite") are purely algebraically described. They are those vector bundles E with satisfy a polynomial equation $f(E) = g(E)$, where $f, g \in \mathbb{N}[T], f \neq g$. (Here in characteristic 0 the category

is semi-simple, so there is no need to introduce the essentially finite bundles [9, Section 2]). However, those equations involve higher rank bundles as well, so it is difficult to cut them down to $M_{dR}(X)$.

3. Characteristic $p > 0$

Let X be a smooth connected quasi-projective variety defined over an algebraically closed field k of characteristic $p > 0$, endowed with a point $a \in X(k)$. Clearly we do not have a notion of topological fundamental group at disposal. But we have the theory of étale fundamental groups.

We denote by $X^{(1)}$ the pull-back of X over the Frobenius of $\operatorname{Spec} k$, and by $F_{X/k} : X \to X^{(1)}$ the Frobenius of X relative to k. The Homs in the category of bundles with a flat connection are linear over $\mathcal{O}_{X^{(1)}}$, so, unless X is proper, there are not finite dimensional k-vector spaces. On the other hand, the category of $\mathcal{O}_X$-coherent $\mathcal{D}_X := \mathcal{D}_{X/k}$-modules is, as in characteristic 0, an abelian, rigid tensor category [13]. The rigidity comes from the fact that if E is a $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module, then it is locally free (Katz, [13, Theorem 1.3, Proof]). Fixing a rational point $a \in X(k)$ defines a fiber functor ω_a to the category of finite dimensional k-vector spaces, by assigning to E its restriction in a. This defines a pro-algebraic group scheme $\pi_1^{\mathrm{alg}}(X, a) := \mathrm{Aut}^{\otimes}(\omega_a)$.

Recall that the Riemann-Hilbert correspondence over $\mathbb{C}$ is between $\mathcal{O}_X$-coherent regular singular $\mathcal{D}_X$-modules and local systems, that is solutions, in the analytic topology, of the linear differential equation in the Zariski topology attached to the $\mathcal{D}_X$-module. Katz in [13, Theorem 1.3] showed an analog statement to the Riemann-Hilbert correspondence in characteristic $p > 0$. As a consequence of Cartier's characterization of p-curvature 0 connections ([16, Theorem 5.1]), if E is a coherent $\mathcal{D}_X$-module, then the associated flat connection is spanned by flat sections, now in the Zariski topology. This defines a bundle $E^{(1)}$ on $X^{(1)}$ together with an isomorphism $(E, \nabla) \cong (F^*E^{(1)}, d \otimes 1_{E^{(1)}})$, where we write $F^*_{X/k}E^{(1)} = \mathcal{O}_X \otimes_{F^{-1}_{X/k}\mathcal{O}_{X^{(1)}}} F^{-1}_{X/k}E^{(1)}$. Then $E^{(1)}$ is precisely the subsheaf of E annihilated by all the differential operators of order $\leq (p-1)$ and is a vector bundle on $X^{(1)}$. In particular, the restriction $\sigma_0 : E \to F^*_{X/k}E^{(1)}$ to the bundles of the isomorphism of flat connections determines uniquely the whole isomorphism. Further, the differential operators of X of order $\geq p$ act on $E^{(1)}$. The subsheaf of sections annihilated by all differential operators of order $\leq (p^2 - 1)$ is a vector bundle $E^{(2)}$ on $X^{(2)}$ and one has an isomorphism of vector bundles $\sigma_1 : E^{(1)} \to F^*_{X^{(1)}/k}E^{(2)}$ etc. A stratified bundle $\mathbb{E} = (E^{(n)}, \sigma_n)_{n \in \mathbb{N}}$ is a sequence of bundles $(E^{(0)} = E, E^{(1)}, E^{(2)}, \ldots)$ together with a sequence of $\mathcal{O}_{X^{(n)}}$-isomorphisms $\sigma_n : E^{(n)} \to F^*_{X^{(n)}/k}E^{(n+1)}$ of bundles. A morphism $\varphi : \mathbb{E} \to \mathbb{E}'$ is a collection $(\varphi_0, \varphi_1, \varphi_2, \ldots)$ where $\varphi_n : E^{(n)} \to E'^{(n)}$ is a bundle map commuting with the σ_i and σ'_i. Katz' theorem (*loc. cit.*) asserts that the functor which assigns to a $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module its underlying stratified bundle as explained above is an

equivalence of categories. This is analog to the Riemann-Hilbert correspondence. However, as it does not involve a stronger topology than the Zariski one, there is no growth condition at infinity of solutions which appears here in the non-proper case. This notion enters later in the sequel.

If $\pi : Y \to X$ is a finite étale cover, then $\pi_* \mathcal{O}_Y$ is an $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module. This defines a surjective homomorphism

$$\pi_1^{\text{alg}}(X, a) \to \pi^{\text{ét}}(X, a) \tag{3.1}$$

of pro-algebraic groups, if one considers $\pi^{\text{ét}}(X, a)$ as a constant pro-finite algebraic group over k. By [8, Proposition 13], this map is the pro-finite completion, as in (2.2).

The analogy with the characteristic 0 theory becomes more involved when one discusses (2.3). Finite étale tame covers (see [17] for a precise and detailed account) define a full subcategory of the category of finite étale covers. This defines the quotient $\pi_1^{\text{ét,tame}}(X, a)$ of $\pi_1^{\text{ét}}(X, a)$. On the other hand, Gieseker [13, Section 3] defines regular singular $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules, assuming X admits a smooth projective compactification $\hat{X}$ such that $\hat{X} \setminus X$ is a strict normal crossings divisor. In [18, Section 3], the concept of a regular singular $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module is defined unconditionally. If $X \subset \hat{X}$ is a partial (i. e. is not necessarily projective) smooth compactification such that $\hat{X} \setminus X$ is a strict normal crossings divisor, regularity with respect to this partial compactification is defined as usual, by assuming the existence of a $\mathcal{O}_{\hat{X}}$-coherent extension of the underlying $\mathcal{O}_X$-locally free sheaf, on which the $\mathcal{D}_X$-action extends to an action of the differential operators which stabilize the ideal sheaf of $\hat{X} \setminus X$. Then the $\mathcal{D}_X$-module is regular singular if it is relatively to all such partial compactifications. This notion coincides with Gieseker's one if one has a good compactificaton. The full subcategory of $\mathcal{O}_X$-coherent regular singular $\mathcal{D}_X$-modules is a sub Tannaka category. This defines a quotient $\pi_1^{\text{alg,rs}}(X, a)$ of $\pi_1^{\text{alg}}(X, a)$. The main theorem of [18, Section 4] asserts that the composite of (3.1) with $\pi_1^{\text{ét}}(X, a) \to \pi_1^{\text{ét,tame}}(X, a)$ factors through $\pi_1^{\text{alg,rs}}(X, a)$

$$\begin{tikzcd}
\pi_1^{\text{alg}}(X, a) \arrow[r] \arrow[d] & \pi^{\text{ét}}(X, a) \arrow[d] \\
\pi_1^{\text{alg,rs}}(X, a) \arrow[r] & \pi^{\text{ét,tame}}(X, a)
\end{tikzcd} \tag{3.2}$$

and that $\pi^{\text{ét,tame}}(X, a)$, while considered as a constant pro-algebraic group over k, is the pro-finite completion of $\pi_1^{\text{alg,rs}}(X, a)$.

Let k be an algebraically closed field of characteristic $p > 0$. So we may raise the following questions in analogy with Proposition 2.1.

Questions 3.1. i) Let $f : X \to Y$ be a morphism between smooth quasi-projective varieties mapping $a \in X(k)$ to $b \in Y(k)$. Assume

$$f_* : \pi_1^{\text{ét,tame}}(X, a) \to \pi_1^{\text{ét,tame}}(Y, b)$$

is an isomorphism, then is

$$f_* : \pi_1^{\text{alg,rs}}(X, a) \to \pi_1^{\text{alg,rs}}(Y, b)$$

an isomorphism as well?

ii) Let X be a smooth connected quasi-projective variety with $a \in X(k)$, such that $\pi_1^{\text{ét,tame}}(X, a)$ is trivial. Then is $\pi_1^{\text{alg,rs}}(X, a)$ trivial as well?

The theory of tame fundamental groups, unfortunately, isn't well developed. Natural properties are now yet known. For example, as far as we are aware of, Künneth formula and the base change property are not known. They are both known for the maximal prime to p quotient of $\pi_1^{\text{ét}}(X, a)$ (see [27] and in it Remarque 5.3 for base change). Topological finite presentation or even topological finite generation are not known (see [26, Théorème 6.1] for finite generation and for X being the complement of a divisor in a smooth projective curve over an algebraically closed field k, but in higher dimension, we do not know the necessary Lefschetz theorems to conclude). A main obstruction to generalize the corresponding results in [30] is the absence of resolution of singularities (see for example [23, Theorem A.15] for the homotopy exact sequence for smooth varieties over an algebraically closed field for the tame fundamental groups under the assumption of resolution of singularities).

Of course we can raise the same questions dropping the tameness assumption.

Questions 3.2. i) Let $f : X \to Y$ be a morphism between smooth quasi-projective varieties mapping $a \in X(k)$ to $b \in Y(k)$. Assume

$$f_* : \pi_1^{\text{ét}}(X, a) \to \pi_1^{\text{ét}}(Y, b)$$

is an isomorphism, then is

$$f_* : \pi_1^{\text{alg}}(X, a) \to \pi_1^{\text{alg}}(Y, b)$$

an isomorphism as well?

ii) Let X be a smooth connected quasi-projective variety with $a \in X(k)$, such that $\pi_1^{\text{ét}}(X, a)$ is trivial. Then is $\pi_1^{\text{alg}}(X, a)$ trivial as well?

If X is smooth, then $X \setminus \Sigma$ has the same $\pi_1^{\text{ét}}$, resp. π_1^{alg} as X when Σ has codimension ≥ 2. So Questions 3.2 reduce to X, Y projective smooth in i), and X projective smooth in ii) if we start with $X \setminus \Sigma_X$ and $Y \setminus \Sigma_Y$, with Σ_X, Σ_Y of codimension ≥ 2. On the other hand, by blowing up several times and removing divisors of a smooth projective variety with $\pi_1^{\text{ét}}(X, a) = 0$, one easily constructs examples of non-proper smooth varieties X^0 with $\pi_1^{\text{ét}}(X^0, a) = 0$, with a nice normal compactification such that the locus at infinity has codimension ≥ 2, but the compactification is not smooth (see [18]). So in absence of resolution

of singularities, and in view of the difficulty to find interesting examples, this is meaningful to ask, even if in characteristic 0, the answer is negative, as one sees for example on the affine line. Indeed, any non-zero closed 1 form ω on $\mathbb{A}^1$ defines a non-trivial connection $d+\omega$ on $\mathcal{O}_{\mathbb{A}^1}$. But, in characteristic $p>0$, $\pi_1^{\text{ét}}(\mathbb{A}^1,0)$ is highly non-trivial.

When X is smooth projective, Question 3.1, or, equivalently, Question 3.2 is Gieseker's conjecture [13, p. 8]. In analogy with Corollary 2.2, Gieseker [13], *loc. cit.* further raised the following questions.

Questions 3.3. Let X be a smooth connected projective variety with $a \in X(k)$.

i) Does every irreducible $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module have rank 1 if and only if the commutator $[\pi_1^{\text{ét}}(X,a),\pi_1^{\text{ét}}(X,a)]$ is a pro-p-group?
ii) Is every $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module a direct sum of rank 1 ones if and only if $\pi_1^{\text{ét}}(X,a)$ is abelian without nontrivial p-power quotient?

Finally, assuming X projective smooth, while we have at disposal Langer's quasi-projective moduli varieties of Gieseker semi-stable bundles [22], we do not have any analog to Simpson's quasi-projective moduli space $M_{dR}(X)$ and of $M_{\text{Betti}}(X)$.

4. Results

The aim of this section is to describe the answers we have to Questions 3.1, 3.2, 3.3. Unfortunately, they are only partial answers.

4.1. **Question 3.2, ii).** It has a positive answer when X is projective (see [10, Theorem 1.1]), this is the main result. We describe now the analogies between the proof of Proposition 2.1, ii) over $\mathbb{C}$ and the proof of [10], *loc. cit.*

Over $\mathbb{C}$ we were using that the monodromy group of a representation of $\pi_1^{\text{top}}(X,a)$ lies in some $GL(r,A)$ for A a ring of finite type over $\mathbb{Z}[\frac{1}{N}]$, as $\pi_1^{\text{top}}(X,a)$ is finitely generated. This is a statement on the complex local system, which is not easy to transpose on the side of the $\mathcal{D}_X$-modules.

In characteristic $p>0$, we do not have a group of finite type controlling the representation $\pi^{\text{alg}}(X,a)\to GL(r,k)$ coming from Tannaka theory. Going to the side of stratified bundles, we do not have quasi-projective moduli spaces for them either. But, X being projective, a stratified bundle $\mathbb{E}$ is up to isomorphism determined by the underlying vector bundles $(E^{(n)})_{n\geq 0}$ (Katz' theorem [13, Proposition 1.7]). If we can make sure that the $E^{(n)}$ are all μ-stable of slope 0, we can "park" them all on one quasi-projective moduli M. That we may assume that they are μ-stable relies on two facts. Firstly there is a structure theorem asserting that there is a natural number n_0 such that the shifted stratified bundle $(E_{n_0},E_{n_0+1},\ldots),(\sigma_{n_0},\sigma_{n_0+1},\ldots)$ is a successive extension of stratified bundles, each of which with the property that its underlying vector bundles are all μ-stable of slope 0 ([10, Proposition 2.3]. Secondly, extensions of the trivial

stratified bundle by itself form a group, which, again using Katz' theorem, is identified with the group $\varprojlim_F H^1(X, \mathcal{O}_X)$, where F is the absolute Frobenius, which is trivial if $\pi_1^{\text{ét}}(X, a) = 0$. In fact $H^1(X, \mathbb{Z}/p) = 0$ is enough here to conclude ([10, Proposition 2.4]).

Further over $\mathbb{C}$, once one has $\rho : \pi_1^{\text{top}}(X, a) \to GL(r, A)$, we find a closed point $s \in \operatorname{Spec} A$ such that the specialization $\rho \otimes \kappa(s) : \pi_1^{\text{top}}(X, a) \to GL(r, \kappa(s))$ is not trivial if ρ is not trivial.

In characteristic $p > 0$, we consider a model M_R of M over some ring R of finite type over $\mathbb{F}_p$. Even of M is not a fine moduli space, it is better than a coarse moduli space, in particular it gives a moduli interpretation of $\bar{\mathbb{F}}_p$-points of M_R, assuming that X itself has a model X_R over R. We consider the Zariski closure $N \subset M$ of all the moduli points $E^{(n)}$, and specialize N to a closed point s of $\operatorname{Spec} R$ for some R on which it is defined. Then the whole point is to show that some such specialization contains a moduli point V say which is fixed by some power of Frobenius ([10, Theorem 3.14]). This yields a Lang torsor over $X_R \otimes_R s$ by resolving the Artin-Schreier type equation $(F^m - \mathrm{Id})^*(V) = 0$ [21, Satz 1.4]. The theory of specialization of the étale fundamental group [30, X, Théorème 3.8] forces then V to be trivial, a contradiction, unless N is empty. Now, in order to find such a point V, one has to apply Hrushovsky's fundamental result [15, Corollary 1.2] stemming from model theory. This, undoubtedly, is a deeper step than finding over $\mathbb{C}$ an s for which $\rho \otimes \kappa(s)$ is not trivial.

4.2. **Question 3.3.** It has a positive answer. The surjection $\pi_1^{\text{alg}}(X, a) \to \pi_1^{\text{ét}}(X, a)$ together with some classical representation theory of finite p-groups imply the one direction (see [13, Theorem 1.10]): assuming irreducible $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules have rank 1, then $[\pi_1^{\text{ét}}(X, a), \pi_1^{\text{ét}}(X, a)]$ is a pro-p-group and if the category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules is semi-simple, then $\pi_1^{\text{ét}}(X, a)$ is abelian without p-power quotient.

We described the other direction (see [11]). The method is derived from [10], with some new ingredients, which we briefly explain now.

We discuss i). By the discussion of the proof of Question 3.2 ii), we extract the information that if X has a non-trivial $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module of rank ≥ 2, then it also has one of the same rank with finite monodromy [11, Theorem 2.3]. This, by classical theory of representation of p-groups, forbids the commutator of the monodromy group to be a p-group.

We discuss ii). One has to show that the category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules is semi-simple. As irreducible $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules have rank 1, one has to show that there are no non-trivial extension of the trivial $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules by a rank 1 one. To this aim, we have to replace Langer's moduli M in the proof of Question 3.2 ii) by some quasi-projective moduli M' say, to be constructed, of non-trivial extensions of $\mathcal{O}_X$ by line bundles L. The assumption

implies that such an extension of $\mathcal{O}_X$ by a torsion line bundle L splits, and does after specialization to $X_R \otimes_R s$ for a closed point. Hrushovsky theorem as explained before enables one to find split extensions as moduli points of $N' \otimes_R s$, a contradiction.

4.3. **Rank 1.** On X quasi-projective smooth over an algebraic closed field k of characteristic $p > 0$, $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules of rank 1 are always regular singular [13, Theorem 3.3]. So Question 3.1 ii) and Question 3.2 ii) are equivalent for rank 1 objects. It has a positive answer [18, Section 5]. In fact, if one restricts to rank 1, it is enough to assume that $\pi_1^{\text{ét,ab},p'}(X,a) = 0$, the prime to p maximal abelian quotient of $\pi_1^{\text{ét}}(X,a)$. The point is that under this assumption, necessarily units on X are constant. This implies that a rank 1 stratified bundle is entirely determined by the underlying bundles $(L^{(n)})_{n\in\mathbb{N}}$. Then one shows by geometry that the assumption implies that $\mathrm{Pic}(X)$ is a finitely generated abelian group ([18, Section 5]).

4.4. **The affine space.** A simple discussion (led with Lars Kindler) implies that Question 3.1 ii) has positive answer for $X = \mathbb{A}^n$. Indeed by Theorem [13, Theorem 5.3], any flat regular singular $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module is a sum of such rank 1 $\mathcal{D}_X$-modules. If L is such a rank 1 $\mathcal{D}_X$-module, then there is a $\mathcal{O}_X$-locally free extension $\hat{L}$ on $\mathbb{P}^n$ on which the action of $\mathcal{D}_X$ on L extends to an action of $\mathcal{D}_{\mathbb{P}^n}(\log\infty)$. As an algebraic vector bundle, $L = \mathcal{O}_{\mathbb{P}^n}(d\cdot\infty)$, where $\infty = \mathbb{P}^n \setminus X$. The connection $\nabla^{(0)} : \hat{L} \to \Omega^1_{\mathbb{P}^n}(\log\infty)\otimes\hat{L}$ on it is necessarily of the shape $d+A$. Here d is the connection which is uniquely defined on $\mathcal{O}(d\cdot\infty)$ by its restriction to $\mathcal{O}_{\mathbb{P}^n} \hookrightarrow \mathcal{O}(d\cdot\infty)$ given by the section $d\cdot\infty$, where it is defined by $d(1) = 0$. Then $A \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(\log\infty))$, which is 0. So finally $\nabla^{(0)}$ has no poles and is trivial. We repeat this argument for $\hat{L}^{(i)}$ on $(\mathbb{P}^n)^{(i)}$ and conclude that in fact $\hat{L}$ is a $\mathcal{O}_{\mathbb{P}^n}$-coherent $\mathcal{D}_{\mathbb{P}^n}$-module, which is trivial.

5. Some comments and questions

5.1. **Moduli.** On X projective smooth over an algebraic closed field k, it would be very nice to extend Simpson's theory to construct moduli spaces (would they be quasi-projective?) for stratified bundles and for $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules.

Recall that over a field of characteristic 0, a bundle with a flat connection (E,∇) with finite monodromy is uniquely determined by E ([9], *loc. cit.*). It would be very nice to understand how the homeomorphism between $M_{\text{Betti}}(X)$ and $M_{dR}(X)$, followed by the forgetful morphism $(E,\nabla) \mapsto E$ with values in the moduli of bundles, transports points corresponding to finite local systems.

5.2. **From characteristic p to characteristic 0 and vice-versa.** In spite of the strong analogy between Proposition 2.1 ii) for X projective and Question 3.1 ii) for X projective ([10], *loc. cit.*), and between Corollary 2.2 and Question 3.3

([11], *loc. cit.*), we do not have a direct way to go back and forth between characteristic $p > 0$ and characteristic 0. Indeed, a flat connection in characteristic 0 specializes to a flat connection in characteristic $p > 0$, that is to an action of the differential operators of order ≤ 1, but this action does not extend to an action of $\mathcal{D}_X$. Already to request p-curvature 0 (which is equivalent, as already explained, to the existence of $E^{(1)}$) for almost all p should be, according to Grothendieck's p-curvature conjecture, equivalent to the connection in characteristic 0 to have finite monodromy. Assuming Simpson's moduli spaces $M_{dR}(X)$ specialize to moduli spaces of flat connections for almost all p, at least at the level of geometric points, a positive answer to Grothendieck's conjecture, for X projective, would provide some answer to the previous question. In the present state of knowledge, we know the well behavior of moduli points corresponding to finite monodromy only in equal characteristic 0 for de Rham moduli ([1]) and $p > 0$ for moduli of semi-stable sheaves or, in absence of moduli, for families of stratified bundles, assuming the monodromy groups have order prime to p ([12]).

5.3. **Question 3.1 ii).** Of course, all unanswered questions in 3.2, 3.1 are of interest, that is all relative cases and the absolute non-proper case. Focusing on this one, even if one assumes that X has a compactification $\hat{X}$ such that $\hat{X} \setminus X$ is a strict normal crossings divisor, the method of proof in the projective case [10] can not be overtaken as such. The bundles $\hat{E}^{(n)}$ on $\hat{X}$ do not all have the same Chern classes, so we can not park them all in one moduli space.

5.4. **Finiteness.** A natural question which comes from [3], and from [11], is whether or not, if X is smooth projective over an algebraically closed field, such that the profinite completion map $\pi_1^{\text{alg}}(X, a) \to \pi_1^{\text{ét}}(X, a)$ is an isomorphism, i. e. all $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules have finite monodromy, $\pi_1^{\text{ét}}(X, a)$ itself is finite.

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